$$S = \frac{Z_P}{Z_e}$$

- Shape factor is a property of the cross-sectional shape. It is independent of the material properties.

- **Advantages of plastic design :**
  1. Saving of materials over elastic methods resulting in lighter structures.
  2. Realization of uniform and realistic factor of safety for all parts of the structure. In case of elastic method, factor of safety varies.
  3. Design moments can be calculated rapidly.
  4. It gives an idea about the type of collapse and the strength of structure.
  5. In plastic design, balanced section is obtained in a single attempt while in working stress method design process is repeated several times to obtain an optimum solution.
  6. No effect due to change in temperature, settlement of supports, imperfections etc. Design is very popular for the design of beams & portal frame.

- **Disadvantages :**
  1. There is little saving in column design
  2. It is difficult to design for fatigue.
  3. If structure is complicated, it is difficult to obtain collapse load.
  4. Lateral bracing system requirements are more stringent than for elastic design.

7.3 PLASTIC HINGE : **(GTU, Dec. 2010, Dec. 2014)**

A plastic hinge is a zone of yielding due to flexure in a structural member.

Consider a simply supported beam loaded by gradually increasing concentrated load W at mid span. The increase of load causes proportionate increase in maximum bending moment at mid span. Since the rotation 'θ' is proportional to the maximum strain, the moment of resistance is proportional to the maximum stress. The moment rotation relationship is shown in fig. 7.4.

With increase in moment, a stage is reached when the maximum stress in the outermost fibre reaches the yield stress f_y, and the corrosponding moment M_y.

With increase in moment ($M > M_y$) but less than M_P, part of the outer section becomes plastic while the inner portion near the neutral axis still remains elastic. The section is therefore, in elasto-plastic condition.

As the load increases a stage is reached when the entire section becomes plastic and the section is no more in a position to offer any additional moment of resistance and the section simply rotates under constant moment.

At the centre of the area there will be full plasticity over the full depth of the section and the section will behave like a hinge. This is called the **plastic hinge.**

... Lateral bracing system requirements are more stringent than for elastic design.

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At the section of plastic hinge, the restraining moment is equal to plastic moment M_P.

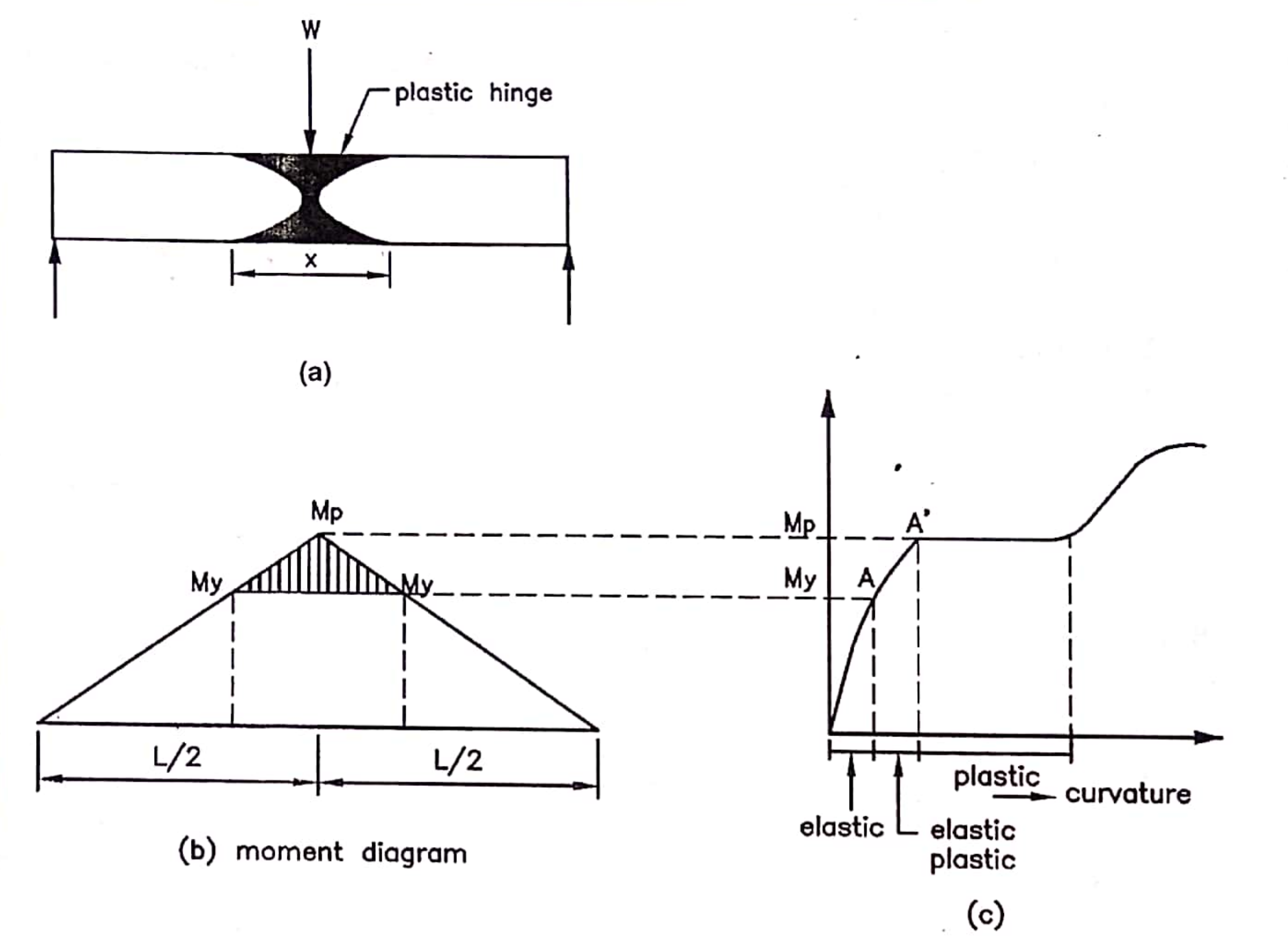


FIG 7.4 PLASTIC HINGE CONCEPT

- **The plastic hinge occur at :**
  - Plastic hinges form in a member at the maximum bending moment locations.
  - However, at the intersections of two members where the bending moment is the same, a hinge forms in the weaker member.
  - Generally, hinges are located at restrained ends, intersection of members and at point loads.
  - At the connection involving change in geometry.
  - At the points of zero shear in a span loaded by u.d.*l*.

The value of moment adjacent to the yield zone is more than the yield moment up to a certain length *x*, of the structural member. This length is known as the yield length or **hinged length.**

(May 2017)

For rectangular section,

$$M_P = f_y \cdot Z_P = f_y \cdot \frac{bh^2}{4}$$

$$M_y = f_y \cdot Z_e = f_y \cdot \frac{bh^2}{6}$$

$$\therefore \quad \frac{M_P}{M_y} = \frac{3}{2} = \text{Shape factor} \qquad \therefore \quad M_y = \frac{2}{3} M_p$$

From fig. 4,

$$\frac{M_p}{L/2} = \frac{M_y}{\frac{L}{2} - \frac{x}{2}}$$

$$\therefore \quad \frac{M_p}{M_y} = \frac{L}{L - x}$$

$$\therefore \quad M_p (L - x) = M_y (L)$$

$$\therefore \quad M_p (L - x) = \frac{2}{3} Mp (L)$$

$$\therefore \quad L - x = \frac{2}{3} L$$

$$\therefore \quad \boxed{x = \frac{L}{3}} \quad \text{... yield length}$$

- **Load factor : (Q) :**

$$Q = \frac{W_c}{W} \text{ or } \frac{W_L}{W} \text{ or } \frac{W_u}{W}$$

Where, Q = Load factor

$W_C = W_L = W_u$ = collapse load or limit load

W = Working load.

- **Collapse Load :** The load at which a sufficient number of plastic hinges are formed in a structure such that a collapse mechanism is created is called the collapse load.

$Q = S \times F$ Where, F = Factor of safety used in elastic design.

$W_C = W \times F$

where,

W = Working load

F = Factor of safety

- **Basic Theorems of plastic Analysis :**

The following three fundamental theorems are used to select the true collapse load.

(i) Static theorem or lower bound theorem

(ii) Kinematic theorem or upper bound theorem

(iii) Uniqueness theorem

(i) Static theorem or lower bound theorem :

It states,

that for a given frame and loading, if there exists any distribution of bending moment throughout the frame which is both safe and statically admissible, with a set of loads W, the value of W must be less than or equal to the collapse load W_C. $W < W_c$. but $M < Mp$.

The static theorem represents the lower limit of the true ultimate load and has a maximum factor of safety. The static theorem satisfy the equilibrium and yield conditions.

(ii) Kinematic theorem or upper bound theorem :

It states,

that for a given frame subjected to a set of loads W, the value of W which is found to correspond to any assumed mechanism will always be greater than or equal to the collapse load W_c.

$W > W_c$. but $M < Mp$.

The kinematic theorem represents an upper limit of the true ultimate load and has a smaller factor of safety. The kinematic theorem satisfy the equilibrium and continuity conditions.

(iii) Uniqueness theorem :

This theorem combines both static as well as kinematic theorems. This theorem states that

7.4 DETERMINATION OF COLLAPSE LOAD FOR SOME STANDARD CASES OF BEAMS.

(1) Simply supported beam carrying a concentrated load W.

Since the beam is statically determinate, $r = 0$

Where, r = degree of redundancy.

Hence only one hinge is required for the collapse of the beam.

We shall solve the problem by two methods i.e. static method and kinematic method.

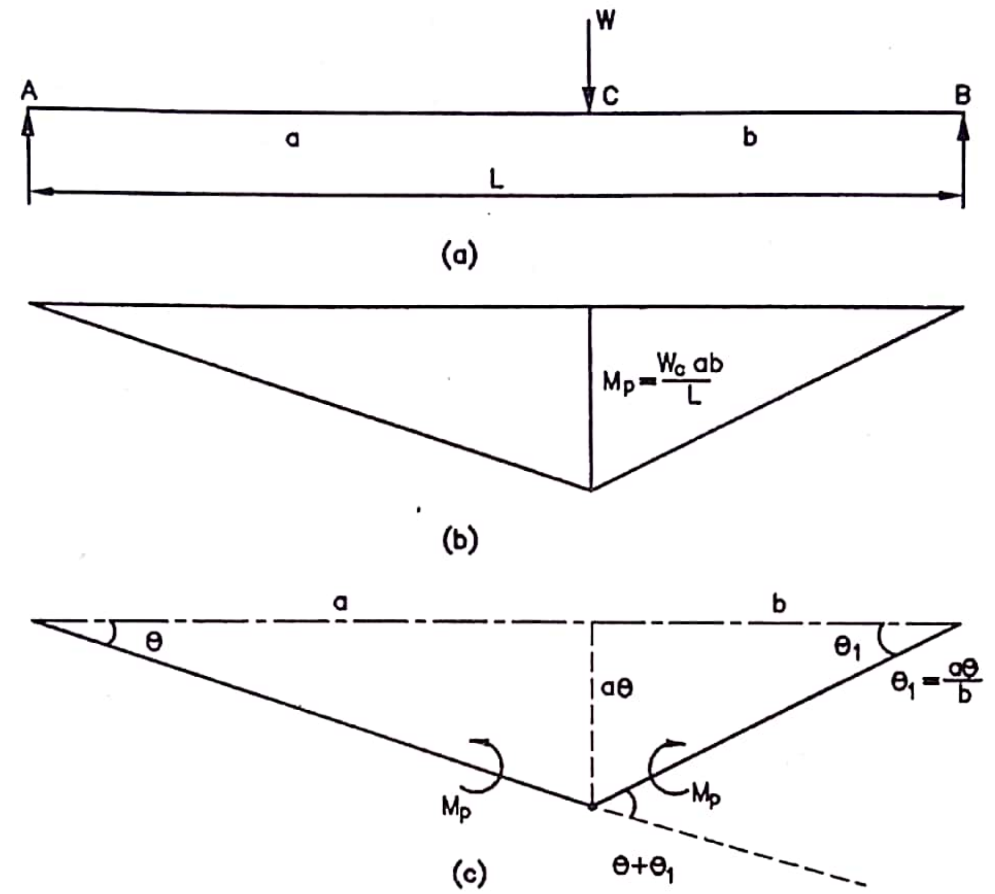


FIG. 7.6

(a) Static method :

Maximum B.M. below point load, $M = \dfrac{Wab}{L}$

When the load is increased to the collapse load W_c, the maximum B.M. will be equal to $\dfrac{W_c\ ab}{L}$

$$\therefore \quad \frac{W_c\ ab}{L} = Mp \text{ or } \boxed{W_c = \frac{M_p \cdot L}{ab}}$$

(b) Kinematic Method :

The collapse mechanism (beam mechanism) is shown in fig. 7.6 (c).

Collapse will occur when a hinge is formed under the load.

Let,

θ = angle of rotation at the left end

$\therefore$ Deflection below the load = $a\theta$

$$\therefore \quad \theta_1 = \frac{a\theta}{b}$$

Rotation of hinge under the action of plastic moment

$$= \theta + \theta_1$$

$$= \theta + \frac{a\theta}{b} = \frac{b\theta + a\theta}{b} = \frac{\theta L}{b}$$

External work done by load $= W_c \cdot a\theta$

Internal work done $= M_p (\theta + \theta_1)$

(Work absorbed by the hinge)

$$= M_p \cdot \frac{\theta L}{b}$$

By the principal of virtual work,

External work done = Internal work done

$$W_c \cdot a\theta = M_p \cdot \frac{\theta L}{b}$$

$$\therefore \quad \boxed{W_c = \frac{M_p \cdot L}{ab}}$$

If the load is acting at the centre of the beam,

$$a = \frac{L}{2}, \; b = \frac{L}{2}$$

$$\therefore \quad W_c = \frac{M_p \cdot L}{\frac{L}{2} \cdot \frac{L}{2}} = \frac{4\, M_p}{L}$$

(2) Simply supported beam carrying U.D.L. :

Let, W = total U.D.L.

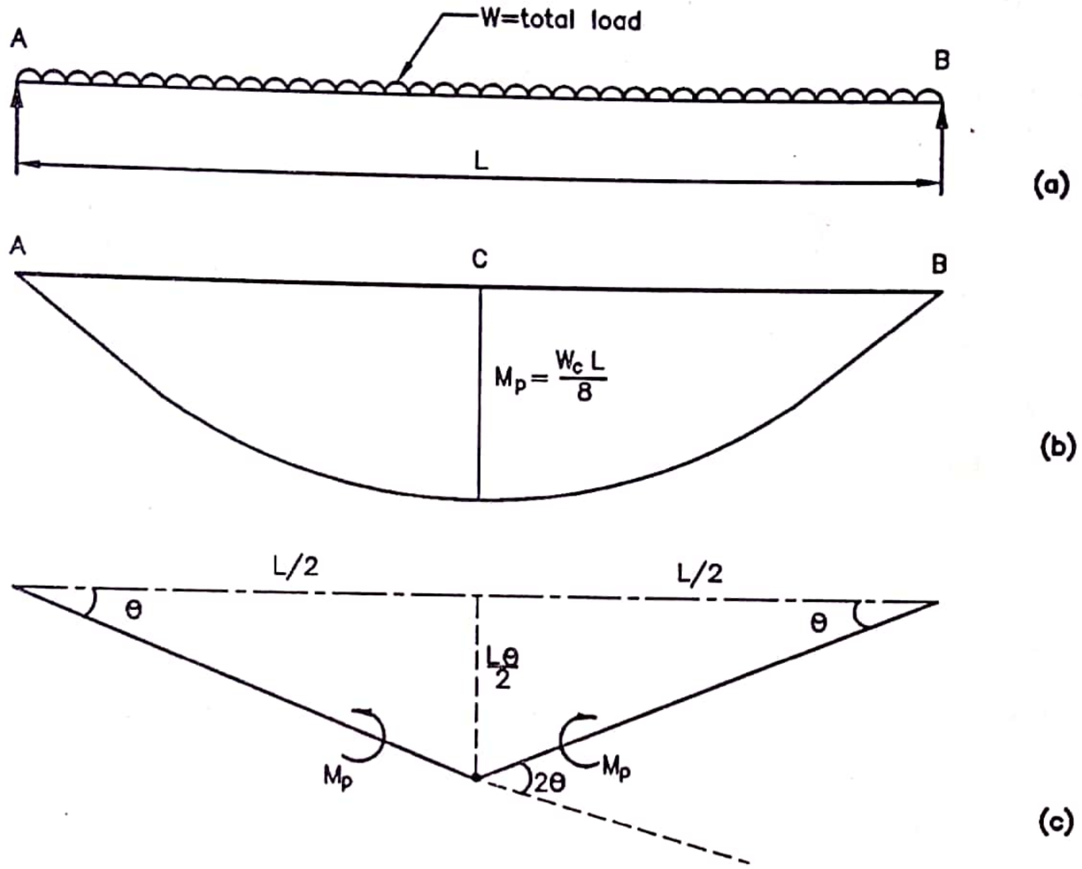


FIG. 7.7

(a) Static method :

Maximum B.M. will occur at the centre of the beam.

$$M = \frac{WL}{8}$$

When the load W is increased to the collapse load W_c, the maximum B.M. will be equal to, $\frac{W_c \cdot L}{8}$

$$\therefore \quad \frac{W_c \cdot L}{8} = M_p \text{ or } \boxed{W_c = \frac{8M_p}{L}}$$

(b) Kinematic Method :

The collapse mechanism is shown in fig. 7.7 (c). Only one hinge is required and it will be formed at the centre of the span.

deflection at the centre of beam $= \frac{L\theta}{2}$

$\therefore$ Average vertical deflection of U.D.L. $= \frac{1}{2} \times \frac{L\theta}{2} = \frac{L\theta}{4}$

External work done by load $= W_c \cdot \frac{L\theta}{4}$

Internal work done $= M_p \cdot 2\theta$

By virtual work principal,

External work done = Internal work done

$$W_c \cdot \frac{L\theta}{4} = M_p \cdot 2\theta$$

$$\therefore \boxed{W_c = \frac{8 M_p}{L}}$$

(3) Propped cantilever with eccentric point load :

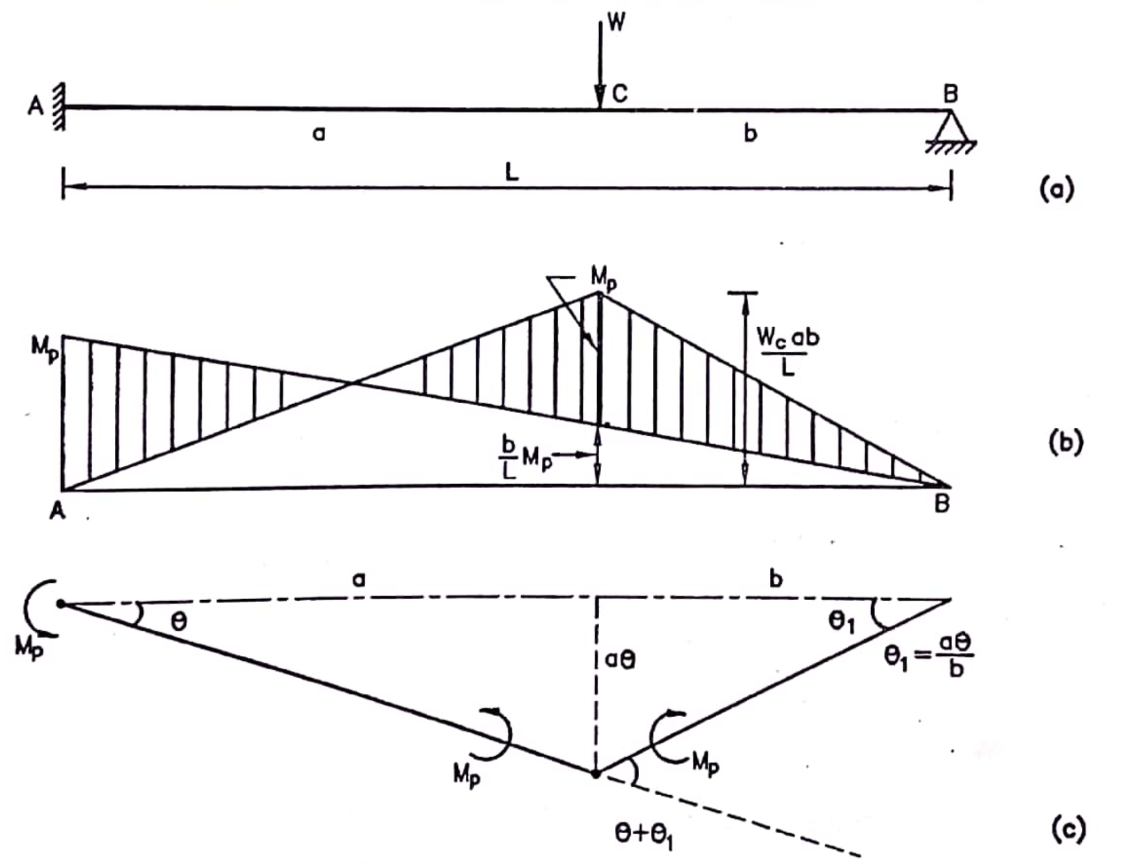


FIG. 7.8

The shape of B.M. diagram during elastic stage will be the same as that shown in fig. 7.8 (b).

Since the static B.M. at C is greater than at A, the plastic hinge will first develop at C and than at A. The structure will then be converted into a mechanism and it will ultimately collapse. During collapse, the moments at both A and C will be M_p.

(a) Static method :

From Fig. 7.8 (b), the equilibrium equation is,

$$\frac{W_c \cdot ab}{L} = M_p + M_p \cdot \frac{b}{L} = M_p \left(\frac{L+b}{L}\right)$$

$$\therefore \quad \boxed{W_c = M_p \frac{(L+b)}{ab}}$$

(b) Kinematic method :

r = 1 (degree of redundancy)

∴ n = r + 1 = 1 + 1 = 2 hinges are required to form a mechanism.

During collapse, one hinge will be formed at fixed end A and the other hinge will be formed at C.

Let, θ = rotation of beam at A

∴ deflection below point load = aθ

$$\therefore \quad \theta_1 = \frac{a\,\theta}{b}$$

The hinge at C will rotate through $\theta + \theta_1$

∴ External work done = $W_c \cdot a\,\theta$

Internal work done = $M_p \cdot \theta + M_p(\theta + \theta_1)$

$$\therefore \quad W_c \cdot a\,\theta = M_p \cdot \theta + M_p\left(\theta + \frac{a\,\theta}{b}\right)$$

$$= M_p \cdot \theta + M_p \cdot \frac{L\,\theta}{b}$$

$$= M_p \cdot \frac{\theta}{b}(L + b)$$

$$\therefore \quad W_c = \frac{M_p}{ab}(L + b)$$

If load act at the centre of beam,

$$a = \frac{L}{2}, \; b = \frac{L}{2}$$

$$\therefore \quad W_c = \frac{M_p\left(L + \frac{L}{2}\right)}{\frac{L}{2} \cdot \frac{L}{2}} = \boxed{\frac{6\,M_p}{L}}$$

(4) Propped cantilever beam with U.D.L. on entire span :

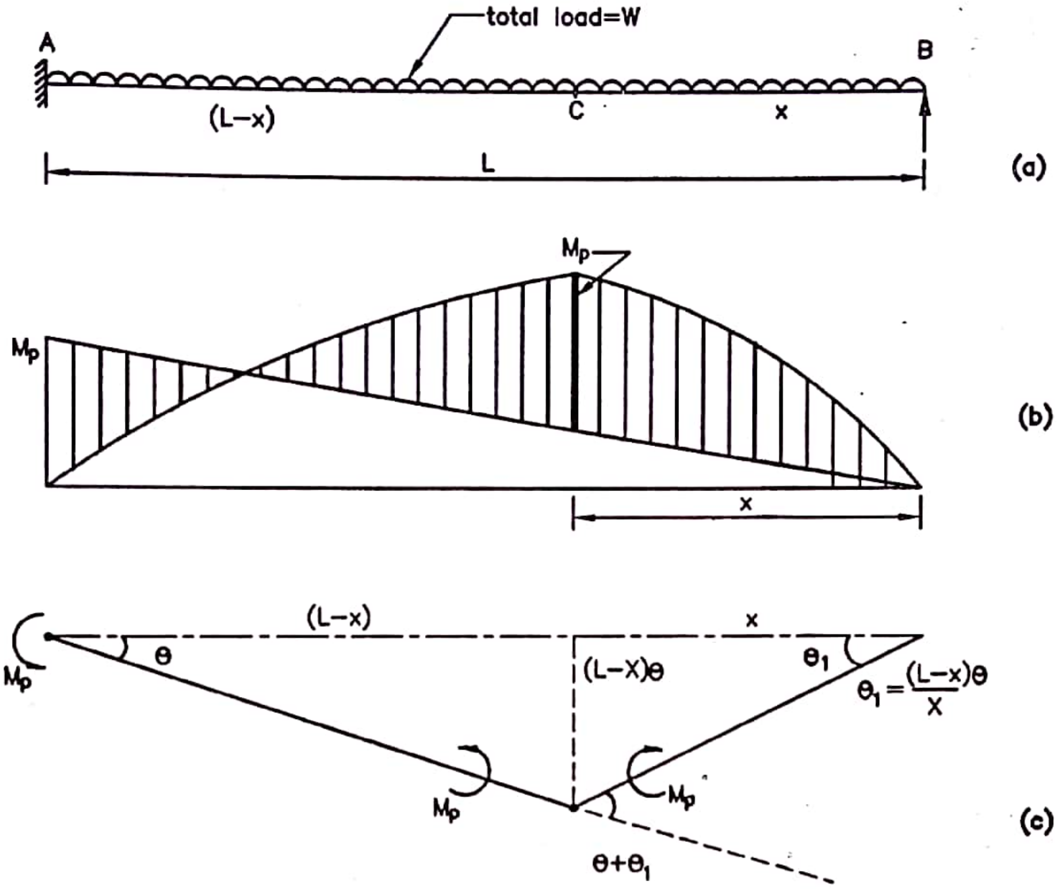


FIG. 7.9

The plastic hinges will be formed at A and C.

The exact location of C is to be determined.

Let M_c be the simply supported B.M. at C, distant x from B.

$$\text{Then } M_c = \frac{W_c x}{2} - \frac{W_c}{L} \cdot \frac{x^2}{2} = \left(M_P + M_P \cdot \frac{x}{L} \right)$$

$$\therefore \quad Mp\left(1 + \frac{x}{L}\right) = \frac{W_c \cdot x}{L}\left(\frac{L}{2} - \frac{x}{2}\right)$$

$$\therefore \quad Mp\,(L + x) = \frac{W_c \cdot x}{2}\,(L - x)$$

$$\therefore \quad Mp = \frac{W_c}{2}\, x \,\frac{(L - x)}{(L + x)} = \frac{W_c}{2}\,\frac{(Lx - x^2)}{(L + x)}$$

For Maxima,

$$\frac{\partial M_P}{\partial x} = 0 = x^2 + 2Lx - L^2$$

$$\therefore \quad x = L\left(\sqrt{2} - 1\right) = \boxed{0.414\ L}$$

(a) Static Method :

The equilibrium equation is,

$$M_p = \frac{W_c}{2} \cdot x \frac{(L - x)}{L + x}$$

$$= \frac{W_c}{2} \frac{0.414\ L\ (L - 0.414\ L)}{L + 0.414\ L}$$

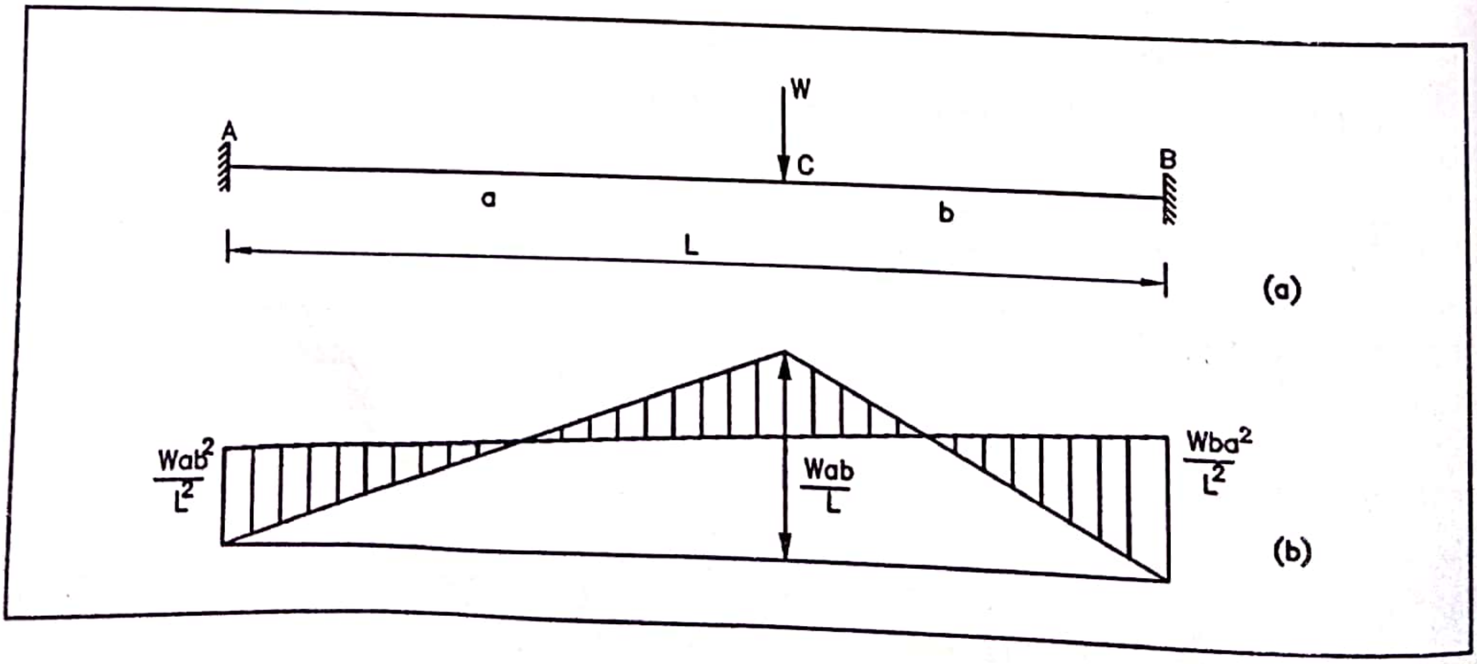
$$M_p = W_c \times L \times 0.0857$$

$$\therefore \quad W_c = \frac{M_p}{L \times 0.08578}$$

$$\therefore \quad \boxed{W_c = 11.656\ \frac{M_p}{L}}$$

(5) Fixed beam carrying an eccentric point load :

Fig. 7.10 (b) shows the B.M. diagram for elastic range. The B.M. at B is greatest. As the load is increased the plastic hinge will first from at B, then at C and finally at A. At this stage the beam is converted into a mechanism and ultimately it will collpse.



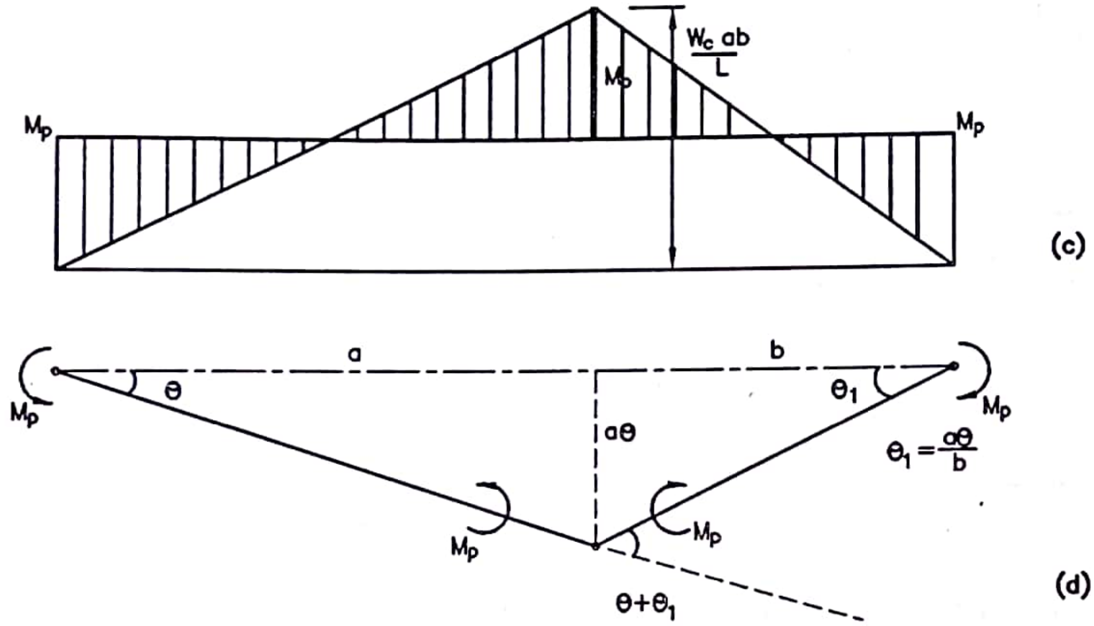


FIG. 7.10

(a) Static Method : From Fig. 7.10 (c)

The equilibrium equation is

$$\frac{W_c \cdot ab}{L} = M_p + M_p = 2\,M_p$$

$$\therefore \quad \boxed{W_c = \frac{2\,M_p \cdot L}{ab}}$$

(b) Kinematic Method :

θ = rotation of beam at A

$\therefore$ deflection below point load = $a\,\theta$

$$\therefore \quad \theta_1 = \frac{a\,\theta}{b}$$

external work done by load = $W_c \cdot a\,\theta$

Internal work absorbed by hinges = $M_p \cdot \theta + M_p\,(\theta + \theta_1) + M_p\,(\theta_1)$

By the principal of virtual work,

$$W_c \cdot a\,\theta = M_p \cdot \theta + M_p\left(\theta + \frac{a\,\theta}{b}\right) + M_p\left(\frac{a\,\theta}{b}\right)$$

$$= M_p \cdot \theta + M_p \cdot \theta\,\frac{L}{b} + M_p\,\frac{a\,\theta}{b}$$

$$= M_p\,\theta\left(1+\frac{L}{b}+\frac{a}{b}\right)$$

$$= M_p\,\theta\,\frac{2L}{b}$$

$$\therefore \quad \boxed{W_c = \frac{2\,M_p \cdot L}{ab}}$$

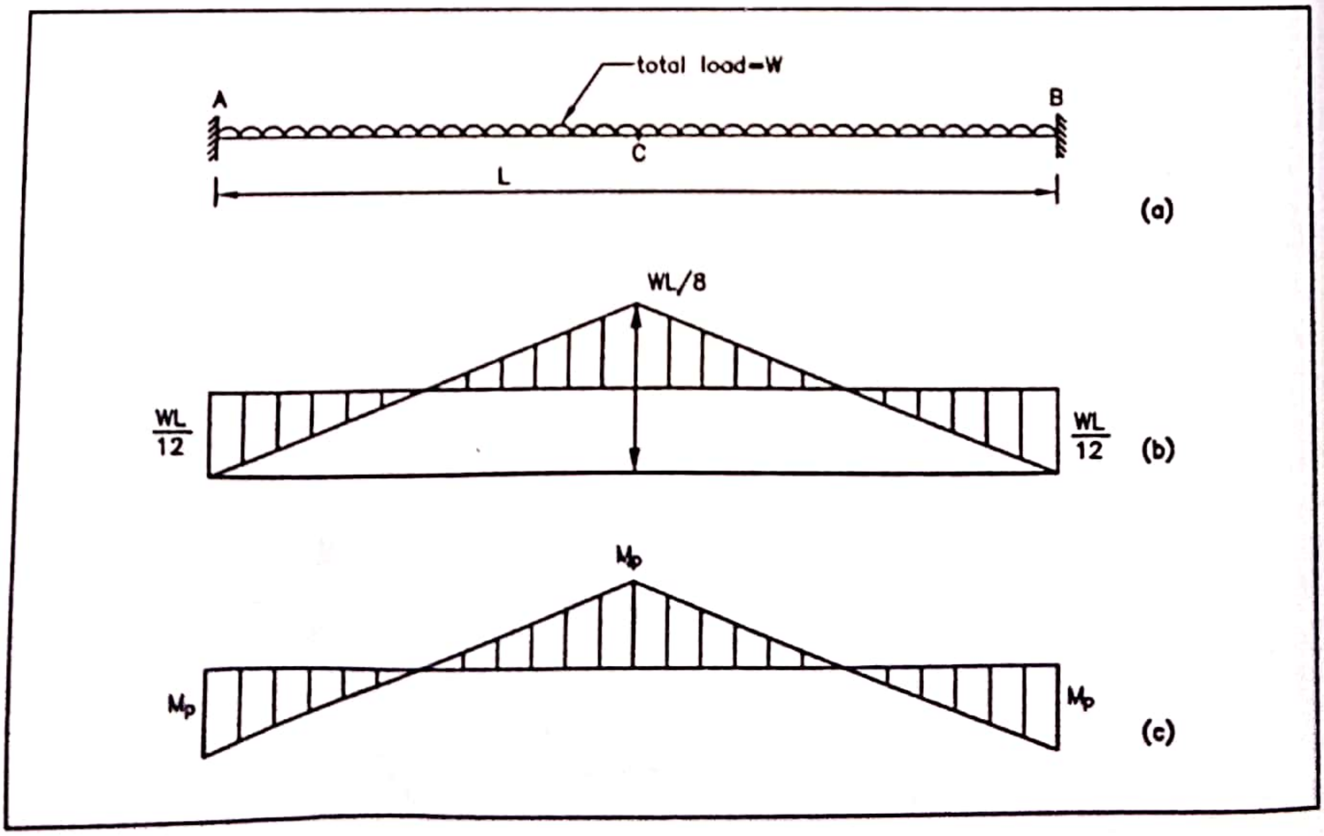
If, the load is placed at the middle of the beam

$$a = \frac{L}{2},\; b = \frac{L}{2}$$

$$\therefore \quad W_c = \frac{2\,M_p \cdot L}{\frac{L}{2}\cdot\frac{L}{2}} = \frac{8\,M_p}{L}$$

6. Fixed beam carrying U.D.L. on entire span

Fig. 7.11 (b) shows the B.M. diagram for elastic range. The B.M. at A and B will be $\frac{WL}{12}$ while that at C will be $\frac{WL}{24}$.



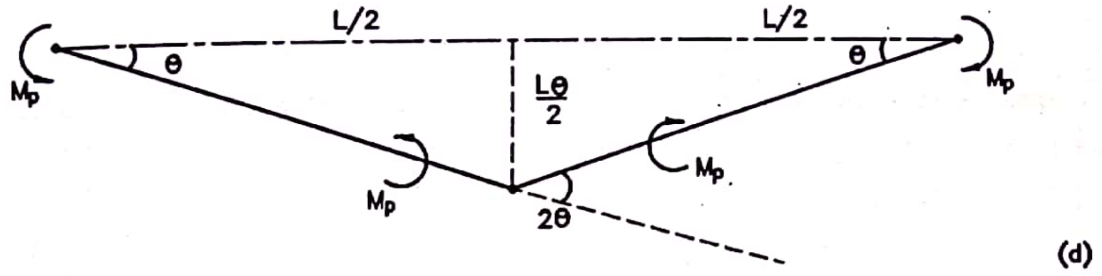


FIG. 7.11

As the load is increased, plastic hinges will first form at A and B, and finally at C. At this stage, the beam will be converted into a mechanism and ultimately it will collapse.

(a) Static Method :

B.M. diagram at plastic stage is shown in fig. 7.11 (c).

The equilibrium equation is,

$$\frac{W_C L}{8} = Mp + Mp = 2Mp$$

$$\therefore \quad \boxed{W_c = \frac{16\ M_p}{L}}$$

(b) Kinematic method :

The rotation at hinges A and B is θ each. Average deflection at centre of beam $= \frac{1}{2} \cdot \frac{L\,\theta}{2}$

External work done by load $= W_c \cdot \left(\frac{1}{2} \cdot \frac{L\,\theta}{2}\right)$

Internal work absorbed by hinges $= M_p \cdot \theta + M_p(2\,\theta) + Mp \cdot \theta$

$= 4\ Mp \cdot \theta$

$$\therefore \quad W_c \frac{L \cdot \theta}{4} = 4\ Mp \cdot \theta$$

$$\therefore \quad \boxed{W_c = \frac{16\ M_p}{L}}$$

7. Three span continuous beam with U.D.L.

Let,

W = total U.D.L on each span

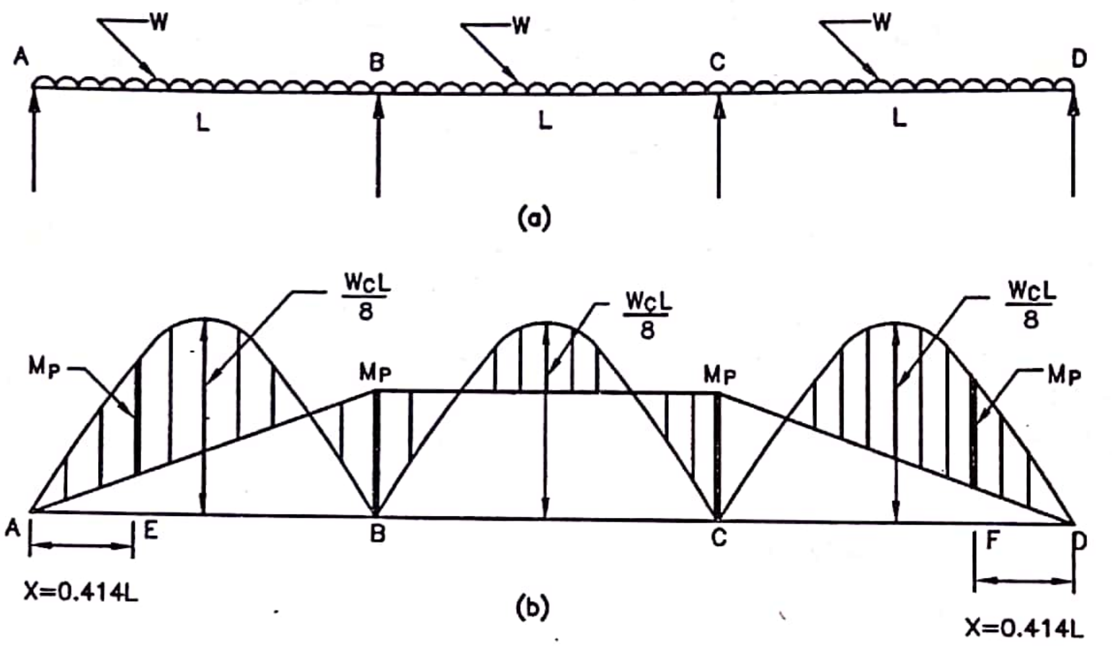


FIG. 7.12

A continuous beam will collapse in the same manner as fixed beam by the formation of three plastic hinges, two at the supports and one between the supports. The failure of one span will result in the failure of the whole strutcure.

Fig. 7.12 (b) shows the B.M. diagram at collapse. During the elastic stage the ordinates of B.M. will be $\frac{WL}{10}$ at the inner supports and $\frac{WL}{8}$ at the mid span. When collapse load is applied, the plastic hinges will form at E, B, C and F and beams AB and CD will collapse. The beam BC can still take more load, but for all practical purposes the continuous beam has been rendered useless.

The span AB and CD may be considered as propped cantilevers with uniformly distributed loading. The collpase load is, therefore $W_c = 11.656\ \frac{M_P}{L}$.

The hinges in the end span form at $x = 0.414$ L from the outer supports.